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On Convergence of Fourier Inverse Transforms for Piecewise Smooth Radial Functions in R^n

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(Dedicated to Professor S.Igari on the occasion of his sixtieth birthday)

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Abstract. For a function $f \in L^p(R^n)$ $(1 \leq p \leq 2)$, we denote by $(S_R f)(x)$ $(R>0)$ the spherical partial sums of Fourier inverse transform of f defined by $\widehat{(S_R f)}(\xi) = \chi_{B(0,R)}(\xi)\hat{f}(\xi)$ and let $f(x)=F(|x|)$ be radial with support in $\{|x| \leqq \alpha\}$ $(\alpha>0)$. In this note, in particular, when $n \geqq 3$, we give a detailed proof of the fact that, for smooth $F \in C^{\ell+2}([0, \alpha])$, $\ell=[(n-3)/2]$, vanishing in a neighborhood of the origin, a necessary and sufficient condition under which we have $\lim_{R\to\infty} (S_R f)(0)=0$ is the validity of $F^{(k)}(\alpha)=0$ for all $k=0,1,\ldots,\ell$. This fact gives a negative answer to the localization problem concerning of $(S_R f)(x)$ for piecewise smooth radial function f.

Let R^n be the n-dimensional Euclidean space and for any $x=(x_1,\ldots,x_n)$, $y=(y_1,\ldots,y_n)$ in R^n we denote $(x,y)=x_1y_1+\cdots+x_ny_n$ and $|x|=\sqrt{(x,x)}$.

For $f \in L^1(R^n)$ we denote the Fourier transform by

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(1) $$\hat{f}(y)=(\sqrt{2\pi})^{-n}\int_{R^n} f(x)\, e^{-i(x,y)}\, dx$$

and the spherical partial sum of its Fourier inverse transform by

(2) $$(S_R f)(x)=(\sqrt{2\pi})^{-n}\int_{|y|\leqq R} \hat{f}(y)\, e^{i(x,y)}\, dy \quad (R>0).$$

It is known that, when $n\geqq 2$, if $f\in C^N(R^n)$, $N=[(n+1)/2]$, and $\frac{\partial^k}{\partial x^k} f\in L^1(R^n)$ for all $k=0,1,\dots,N$, then we have

(3) $$\lim_{R\to\infty} (S_R f)(x)=f(x)$$

for all $x\in R^n$ (W. Pan [2]). Furthermore, even if $f\in L^p(R^n)$ $(1\leqq p\leqq 2)$ is radial with compact support, the localization principle for $(S_R f)(x)$ is not valid (S. Bochner [1]). In this note we consider that, for radial functions with compact support, how smoothness is necessary in order to assure the validity of the localization principle.

In the followings we restrict that f is radial with support in $\{|x|\leqq a\}$ $(a>0)$ and we denote $f(x)$ by $F(|x|)$. For each $m=0,1,\dots$, $F\in C^m([0,a])$ means that $F(t)$ $(0\leqq t\leqq a)$ belongs to the class C^m in $(0,a)$ and that two one-sided limits $F^{(k)}(+0)$ and $F^{(k)}(a-0)$ exist as finite values for all $k=0,1,\dots,m$.

We will write the Fourier inversion formula at $x=0$ as

(4) $$\lim_{R\to\infty} (S_R f)(0)=f(0)$$

and in this note we will give detailed proofs of the following theorems concerning validities of (4) and (3) at $x\neq 0$.

Theorem 1.

(I) When n=1 or 2, if $F\in C^1([0,a])$, then (4) is valid.

(II) When $n\geqq 3$, if $F\in C^{\ell+2}([0,a])$, $\ell=[(n-3)/2]$, then (4) is valid under the condition

(5) $F(a)=F'(a)=\cdots=F^{(\ell)}(a)=0.$

(III) Conversely, for $F\in C^{\ell+2}([0,a])$, $\ell=[(n-3)/2]$, if (5) is not valid, then (4) is not valid. More precisely, denoting $\min\{k \mid 0\leqq k\leqq \ell,\ F^{(k)}(a)\neq 0\}$ by k_0, we have

$$\liminf_{R\to\infty} \frac{(S_R f)(0)-f(0)}{R^{(n-3)/2-k_0}} < 0 < \limsup_{R\to\infty} \frac{(S_R f)(0)-f(0)}{R^{(n-3)/2-k_0}} .$$

Theorem 2. For $n\geqq 1$, $F\in C^2([0,a])$ and $x\neq 0$, we have

$$\lim_{R\to\infty} (S_R f)(x)=\begin{cases} f(x) & (0<|x|<a), \\ f(x)/2 & (|x|=a). \end{cases}$$

Let F be a function in $C([0,a])$ and vanishes in some neighborhood of the origin. Then, according to Theorem 1, when n=1 or 2, (4) is valid if $F\in C^1([0,a])$. On the other hand, in the case of $n\geqq 3$, for $F\in C^{\ell+2}([0,a])$, $\ell=[(n-3)/2]$, (4) is valid if and only if (5) is satisfied. Hence, in this case, the localization principle is not necessarily valid for $F\in C^{\ell+2}$ with compact support.

Theorems 1 and 2 are due to M.A.Pinsky[3] and proofs which we will give in this note by rewriting the proofs in [3] with some calculating devices seem to be somewhat more legible and are more detailed.

§1. Preparations from Bessel functions.

Let $J_\mu(t)$ $(t>0)$ be the Bessel function of order $\mu(>-1)$ and we write $V_\mu(t)=t^{-\mu}J_\mu(t)$. Particularly it is known that

(6) $J_{-1/2}(t)=\sqrt{2/\pi}\ t^{-1/2}\cos t$, $J_{1/2}(t)=\sqrt{2/\pi}\ t^{-1/2}\sin t$.

Now we state some formulas concerning the differentiation and asymptotic formulas for $J_\mu(t)$ (see G.N.Watson[5]).

(i) For $\mu>-1$ we have

(7) $\dfrac{d}{dt}\{t^{-\mu}J_\mu(t)\}=-t^{-\mu}J_{\mu+1}(t)$, i.e. $\dfrac{d}{dt}V_\mu(t)=-t\,V_{\mu+1}(t)$.

Especially $J_0'(t)=-J_1(t)$.

(ii) For $\mu>0$ we have

(8) $\dfrac{d}{dt}\{t^{\mu}J_\mu(t)\}=t^{\mu}J_{\mu-1}(t)$.

(iii) For $\mu>-1$ we have

(9) $J_\mu(t)=(2^\mu\Gamma(\mu+1))^{-1}\,t^\mu\,\{1+O(t^2)\}$ $(t\to+0)$.

Hence $V_\mu(0)=(2^\mu\Gamma(\mu+1))^{-1}$ is reasonable.

(iv) For $\mu \geqq -1/2$ we have

$$(10)\quad J_{\mu}(t)=\sqrt{2/\pi}\ t^{-1/2}\ \{\cos(t-\pi\mu/2-\pi/4)+O(1/t)\}\quad (t\to\infty).$$

For any radial function $f(\cdot)=F(|\cdot|)\in L^1(R^n)$ with $\mathrm{supp}(F)\subset[0, a]$, (1) and (2) can be expressed in terms of Bessel functions as follows.

$$(11)\quad \hat{f}(y)=\int_0^a F(t)\ t^{n-1}\ V_{(n-2)/2}(|y|t)\ dt$$

and we denote it by $\hat{F}_n(|y|)$ in order to emphasize it to be of dimension n.

Further

$$(12)\quad (S_R f)(x)=\int_0^R \hat{F}_n(r)\ r^{n-1}\ V_{(n-2)/2}(|x|r)\ dr$$

and similarly we denote it by $(S_R^{(n)} f)(|x|)$.

At x=0, since

$$V_{(n-2)/2}(0)=(2^{(n-2)/2}\ \Gamma(n/2))^{-1}=(\sqrt{2\pi})^{-n}\omega_{n-1}\ ,$$

where $\omega_{n-1}=2(\sqrt{\pi})^n(\Gamma(n/2))^{-1}$ is the surface area of unit sphere in R^n , then from (11) and (12) we can write as

$$(S_R^{(n)} f)(0)=(\sqrt{2\pi})^{-n}\omega_{n-1}\int_0^R \hat{F}_n(r)\ r^{n-1}\ dr$$

$$=(\sqrt{2\pi})^{-n}\omega_{n-1}\int_0^R \{\int_0^a F(t)\ t^{n-1}\ V_{(n-2)/2}(rt)\ dt\ \}\ r^{n-1}\ dr$$

$$(13)\quad =(\sqrt{2\pi})^{-n}\omega_{n-1}\int_0^a F(t)\ t^{n-1}\ D_R^{(n)}(t)\ dt\ ,$$

where $D_R^{(n)}(t)$ is defined by

$$D_R^{(n)}(t)=\int_0^R V_{(n-2)/2}(rt)\ r^{n-1}\ dr=(\sqrt{2\pi})^{-n}\int_{|y|\leqq R} e^{i(x,y)}\ dy$$

with $|x|=t$, which is called the Dirichlet kernel. This kernel can be expressed in terms of Bessel functions by making use of the integration by substitution and (8) as

$$D_R^{(n)}(t)=t^{-n}\int_0^{Rt} r^{n/2} J_{n/2-1}(r)\,dr=t^{-n}\,(Rt)^{n/2} J_{n/2}(Rt)$$

$$=R^n\,(Rt)^{-n/2} J_{n/2}(Rt)=R^n\,V_{n/2}(Rt). \tag{14}$$

We will use (13),(14) in the proof of Theorem 1 and use (11),(12) in the proof of Theorem 2.

§2. Lemmas.

Lemma 1. For the Dirichlet kernel $D_R^{(n)}(t)$ $(t\neq 0)$ in (14) we have the followings.

(i) For $n\geqq 1$,

$$D_R^{(n)}(t)=\sqrt{2/\pi}\;R^{(n-1)/2}\,t^{-(n+1)/2}\,\{\sin(Rt-(n-1)\pi/4)+O(1/R)\}\quad (R\to\infty).$$

(ii) For $n\geqq 3$,

$$D_R^{(n)}(t)=-\frac{1}{t}\frac{d}{dt}D_R^{(n-2)}(t).$$

Proof.

(i) By (10) we have for $R\gg 1$,

$$D_R^{(n)}(t)=R^n\,V_{n/2}(Rt)$$

$$=R^n\,\sqrt{2/\pi}\,(Rt)^{-n/2-1/2}\{\cos(Rt-n\pi/4-\pi/4)+O(1/R)\}$$

$$=\sqrt{2/\pi}\;R^{(n-1)/2}\,t^{-(n+1)/2}\,\{\sin(Rt-(n-1)\pi/4)+O(1/R)\}.$$

(ii) If we use (7) in $D_R^{(n-2)}(t)=R^{n-2}\,V_{(n-2)/2}(Rt)$, then we have

$$\frac{d}{dt}D_R^{(n-2)}(t)=R^{n-2}\frac{d}{dt}V_{n/2-1}(Rt)=R^{n-2}(-1)\,Rt\,V_{n/2}(Rt)\,R$$
$$=-t\,R^n\,V_{n/2}(Rt)=-t\,D_R^{(n)}(t).$$

Lemma 2. If $F\in C^1([0,\alpha])$, then on $\widehat{F}_n(r)$ in (11) the followings are valid.

(i) For $n\geqq 1$,

$$\widehat{F}_n(r)=O(r^{-(n+1)/2})\quad(r\to\infty).$$

(ii) For $n\geqq 3$,

$$\widehat{F}_n(r)=-\frac{1}{r}\frac{d}{dr}\widehat{F}_{n-2}(r)\quad(r\neq 0).$$

Proof.

In order to prove (i) we note that by making use of (8) and the integration by parts we have

$$\widehat{F}_n(r)=\int_0^{\alpha}F(t)\,t^{n-1}\,V_{n/2-1}(rt)\,dt=r^{-n+1}\int_0^{\alpha}F(t)\,(rt)^{n/2}\,J_{n/2-1}(rt)\,dt$$

$$=r^{-n}\int_0^{\alpha}F(t)\,\frac{d}{dt}\{(rt)^{n/2}J_{n/2}(rt)\}\,dt$$

$$=r^{-n}\{F(\alpha)(r\alpha)^{n/2}J_{n/2}(r\alpha)-\int_0^{\alpha}F'(t)\,(rt)^{n/2}\,J_{n/2}(rt)\,dt\}$$

$$=F(\alpha)\,\alpha^{n/2}\,r^{-n/2}\,J_{n/2}(r\alpha)-r^{-n/2}\int_0^{\alpha}F'(t)\,t^{n/2}\,J_{n/2}(rt)\,dt.$$

By asymptotic formulas (9) and (10) we get

the first term$=O(r^{-(n+1)/2})\quad(r\to\infty)$

and since $F'\in C([0,\alpha])$ we get

$$\text{the second term} = -r^{-n/2} \left\{ \int_0^{1/r} + \int_{1/r}^{a} \right\} F'(t)\, t^{n/2} J_{n/2}(rt)\, dt$$

$$= O\left(r^{-n/2} \int_0^{1/r} t^{n/2}(rt)^{n/2}\, dt\right) + O\left(r^{-n/2} \int_{1/r}^{a} t^{n/2}(rt)^{-1/2}\, dt\right)$$

$$= O\left(\int_0^{1/r} t^{n}\, dt\right) + O\left(r^{-(n+1)/2} \int_{1/r}^{a} t^{(n-1)/2}\, dt\right)$$

$$= O(r^{-(n+1)}) + O(r^{-(n+1)/2}) = O(r^{-(n+1)/2}) \quad (r \to \infty).$$

Thus (i) is proved.

In order to prove (ii), if we note that by (7)

$$\frac{d}{dt} V_{(n-4)/2}(rt) = -rt\, V_{(n-4)/2+1}(rt)\, r = -r^2 t\, V_{(n-2)/2}(rt)$$

and so

$$V_{(n-2)/2}(rt) = -\frac{1}{r^2 t} \frac{d}{dt} V_{(n-4)/2}(rt) = -\frac{1}{rt^2} \frac{d}{dr} V_{(n-4)/2}(rt),$$

then we get

$$\hat{F}_n(r) = \int_0^{a} F(t)\, t^{n-1} V_{(n-2)/2}(rt)\, dt$$

$$= -\frac{1}{r} \frac{d}{dr} \int_0^{a} F(t)\, t^{n-3} V_{(n-4)/2}(rt)\, dt$$

$$= -\frac{1}{r} \frac{d}{dr} \int_0^{a} F(t)\, t^{(n-2)-1} V_{((n-2)-2)/2}(rt)\, dt = -\frac{1}{r} \frac{d}{dr} \hat{F}_{n-2}(r).$$

Lemma 3 (The Hankel inversion formula).

For any function $G(t)$ $(t>0)$ which belongs to $L^1(0,\infty)$ and is of bounded variation in a neighborhood of a point $t=\rho\,(>0)$, we have for $\mu \geqq -1/2$,

$$\lim_{R\to\infty} \int_0^{R} \left\{ \int_0^{\infty} G(t)(rt)^{1/2} J_{\mu}(rt)\, dt \right\} (\rho r)^{1/2} J_{\mu}(\rho r)\, dr$$

$$=\{G(\rho+0)+G(\rho-0)\}/2.$$

This fact is mentioned e.g. in I.Sneddon[4, p.52] or in G.N.Watson[5, p.456].

§3. The proof of Theorem 1.

First we will prove (I) in the cases of n=1 and n=2 and next we will reduce the proof of (II) in $n\geqq 3$ to the result (I).

(I) Let $F\in C^1([0,a])$. In the case of n=1, (13) is

$$(S_R^{(1)}f)(0)=(\sqrt{2\pi})^{-1}\omega_0\int_0^a F(t)\ D_R^{(1)}(t)\ dt$$

and since by (14) and (6)

$$D_R^{(1)}(t)=R\ V_{1/2}(Rt)=R\ (Rt)^{-1/2}\ \sqrt{2/\pi}\ (Rt)^{-1/2}\ \sin Rt=\sqrt{2/\pi}\ \frac{\sin Rt}{t}$$

and $\omega_0=2$, so we have

$$(S_R^{(1)}f)(0)=\frac{2}{\pi}\int_0^a F(t)\ \frac{\sin Rt}{t}\ dt.$$

Now as

$$\lim_{R\to\infty}\frac{2}{\pi}\int_0^a \frac{\sin Rt}{t}\ dt=\lim_{R\to\infty}\frac{2}{\pi}\int_0^{Ra}\frac{\sin t}{t}\ dt=1$$

and $\{F(t)-F(0)\}/t=O(1)$ from $F\in C^1([0,a])$, so we get by the Riemann-Lebesgue theorem

$$\lim_{R\to\infty}\{(S_R^{(1)}f)(0)-f(0)\}=\lim_{R\to\infty}\frac{2}{\pi}\int_0^a\frac{F(t)-F(0)}{t}\sin Rt\ dt=0.$$

In the case of n=2, since again by (13) and (14)

$$(S_R^{(2)}f)(0)=(\sqrt{2\pi})^{-2}\omega_1\int_0^a F(t)\ t\ D_R^{(2)}(t)\ dt,$$

$$D_R^{(2)}(t)=R^2\ V_1(Rt)=R\ t^{-1}\ J_1(Rt),$$

and $\omega_1=2\pi$, so we have

$$(S_R^{(2)} f)(0)=\int_0^{\alpha} F(t)\ R\ J_1(Rt)\ dt.$$

Now since, by making use of (7) especially $J_0'(t)=-J_1(t)$ and (10),

$$\lim_{R\to\infty}\int_0^{\alpha} RJ_1(Rt)dt=-\lim_{R\to\infty}\{J_0(R\alpha)-J_0(0)\}=J_0(0)=1,$$

so we have

$$\lim_{R\to\infty}\{(S_R^{(2)} f)(0)-f(0)\}=\lim_{R\to\infty}\int_0^{\alpha} G(t)RJ_1(Rt)dt,$$

where $G(t)=F(t)-F(0)$. Integrating by parts we have

$$\int_0^{\alpha} G(t)RJ_1(Rt)dt=\int_0^{\alpha} G(t)\ \frac{d}{dt}\{-J_0(Rt)\}dt$$

$$=-G(\alpha)J_0(R\alpha)+\int_0^{\alpha} G'(t)J_0(Rt)dt.$$

We note that the first term is $o(1)$ as $R\to\infty$ by (10). Moreover it can be shown that the second term is $o(1)$ also as $R\to\infty$ as follows. Since $G'(t)\in C([0,\alpha])$, so $G'(t)$ can be uniformly approximated by algebraic polynomials in $[0,\alpha]$. That is, for any given $\varepsilon>0$, there exists $P(t)=\sum_{k=0}^{N} c_k t^k$ such that $|G'(t)-P(t)|<\varepsilon$ for all $t\in[0,\alpha]$. Then

$$\text{the second term}=\int_0^{\alpha}\{G'(t)-P(t)\}J_0(t)dt+\sum_{k=0}^{N} c_k\int_0^{\alpha} t^k J_0(Rt)dt$$

$$=I+\sum_{k=0}^{N} c_k I_k$$

say. Because $J_0(Rt)=O(1)$, we have

$$|I|\leqq\varepsilon\int_0^{\alpha}|J_0(Rt)|dt=O(\varepsilon)$$

uniformly in R. For each $k=0,1,\dots,N$, by (9) and (10) we have

$$I_k=\left(\int_0^{1/R}+\int_{1/R}^{a}\right) t^k J_0(Rt)dt$$

$$=O\left(\int_0^{1/R} t^k\, dt\right)+O\left(\int_{1/R}^{a} t^k\, (Rt)^{-1/2}\, dt\right)=O(R^{-(k+1)})+O(R^{-1/2})$$

$$=O(R^{-1/2})=o(1) \quad (R\to\infty).$$

Thus we get that the second term is $o(1)$ as $R\to\infty$ and (I) is proved.

To prove (II), let $n\geqq 3$. Using Lemma 1(ii) and integrating by parts in (13),

we have

$$(S_R^{(n)}f)(0)=\frac{\omega_{n-1}}{(\sqrt{2\pi})^n}\int_0^{a} F(t)t^{n-1}\left(-\frac{1}{t}\frac{d}{dt}D_R^{(n-2)}(t)\right)dt$$

$$=-\frac{\omega_{n-1}}{(\sqrt{2\pi})^n}\int_0^{a} F(t)\ t^{n-2}\ \frac{d}{dt}D_R^{(n-2)}(t)\ dt$$

$$(15)\quad =-\frac{\omega_{n-1}}{(\sqrt{2\pi})^n}F(a)a^{n-2}D_R^{(n-2)}(a)+$$

$$+\frac{\omega_{n-1}}{(\sqrt{2\pi})^n}\int_0^{a} D_R^{(n-2)}(t)\ \frac{d}{dt}\{F(t)t^{n-2}\}\ dt.$$

Here we note that the following recurrence formula is valid;

$$(16)\quad \frac{\omega_{n-1}}{(\sqrt{2\pi})^n}=\frac{1}{n-2}\frac{\omega_{n-3}}{(\sqrt{2\pi})^{n-2}}=\frac{1}{(n-2)!!}\begin{cases}\dfrac{\omega_0}{\sqrt{2\pi}} & (n:\text{odd}),\\[2mm] \dfrac{\omega_1}{(\sqrt{2\pi})^2} & (n:\text{even}),\end{cases}$$

where $\omega_0=2$ and $\omega_1=2\pi$. If in the case of odd n we write $n=2N+1$ $(N\geqq 1)$ and in the case of even n we write $n=2N+2$ $(N\geqq 1)$, then $\ell=[(n-3)/2]=N-1$ in both of the cases.

To repeat the integration by parts in (15), for our radial function $f(x)=F(|x|)$ with $F \in C^{\ell+2}([0, a])$ we introduce the sequence of functions $\{f_m(t)\}_{m=0, 1, \ldots, N-1}$ such as

$$f_0(t)=t^{n-2} F(t),$$
$$f_m(t)=\frac{1}{t} \frac{d}{dt} f_{m-1}(t) \quad (m=1, 2, \ldots, N-1).$$

Then we find that $f_m(t)$ can be expressed as

$$(17) \quad f_m(t)=(n-2)(n-4)\cdots(n-2m)\, t^{n-2-2m} F(t)+\sum_{1 \leq i, j \leq m} c_{ij}\, t^{n-2-2m+i} F^{(j)}(t)$$

and

$$(18) \quad f_m(t)=t^{n-2-m} F^{(m)}(t)+\sum_{0 \leq i, j \leq m-1} c_{ij}\, t^{n-2-m-(i+1)} F^{(j)}(t).$$

We note that since exponents of t in (17) and (18) are greater than $n-2-2m \geqq$ $\geqq n-2-2(N-1)=n-2N \geqq 1$, so $f_m(0)=0$.

Repeating the integration by parts $(N-1)$times in (15) by making use of Lemma 1(ii) we get

$$(S_R^{(n)} f)(0)$$
$$=-\frac{\omega_{n-1}}{(\sqrt{2\pi})^n} f_0(a) D_R^{(n-2)}(a)+\frac{\omega_{n-1}}{(\sqrt{2\pi})^n} \cdot \int_0^a D_R^{(n-2)}(t) \frac{d}{dt} f_0(t)\, dt$$
$$=-\frac{\omega_{n-1}}{(\sqrt{2\pi})^n} f_0(a) D_R^{(n-2)}(a)-\frac{\omega_{n-1}}{(\sqrt{2\pi})^n} \int_0^a \frac{d}{dt} D_R^{(n-4)}(t)\, f_1(t)\, dt$$
$$=-\frac{\omega_{n-1}}{(\sqrt{2\pi})^n} \sum_{k=1}^{2} f_{k-1}(a) D_R^{(n-2k)}(a)$$
$$+\frac{\omega_{n-1}}{(\sqrt{2\pi})^n} \int_0^a D_R^{(n-4)}(t) \frac{d}{dt} f_1(t)\, dt$$

$$(19)\quad = -\frac{\omega_{n-1}}{(\sqrt{2\pi})^n} \sum_{k=1}^{N} f_{k-1}(\alpha) D_R^{(n-2k)}(\alpha) + \frac{\omega_{n-1}}{(\sqrt{2\pi})^n} \int_0^{\alpha} D_R^{(n-2N)}(t) \frac{d}{dt} f_{N-1}(t)\, dt.$$

From (17) we note that

$$(20)\quad \frac{d}{dt} f_{N-1}(t) = (n-2)(n-4)\cdots(n-2N) t^{n-2N-1} F(t) + \sum_{1 \leq i,\, j \leq N} c_{ij}\, t^{n-2N-1+i} F^{(j)}(t).$$

When $n=2N+1$ $(N \geqq 1)$, by (20) we have

$$\frac{d}{dt} f_{N-1}(t) = (2N-1)!!F(t) + \sum_{1 \leq i,\, j \leq N} c_{ij}\, t^i\, F^{(j)}(t) = G_N(t)$$

say, and we set $g(x)=G_N(|x|)$. In this case since by (16) we have

$$\frac{\omega_{n-1}}{(\sqrt{2\pi})^n} = \frac{1}{(2N-1)!!} \frac{\omega_0}{\sqrt{2\pi}},$$

so we can rewrite (19) as

$$(21)\quad (S_R^{(n)} f)(0) = -\frac{\omega_{n-1}}{(\sqrt{2\pi})^n} \sum_{k=1}^{N} f_{k-1}(\alpha) D_R^{(n-2k)}(\alpha) + \frac{1}{(2N-1)!!} \frac{\omega_0}{\sqrt{2\pi}} \int_0^{\alpha} D_R^{(1)}(t)\, G_N(t)\, dt.$$

The assumption $F \in C^{\ell+2}([0, \alpha]) = C^{N+1}([0, \alpha])$ implies $G_N \in C^1([0, \alpha])$. So from

(I) with n=1 we have

$$\lim_{R\to\infty} \text{the second term of (21)} = \frac{1}{(2N-1)!!} \lim_{R\to\infty} (S_R^{(1)} g)(0) = \frac{1}{(2N-1)!!} g(0)$$
$$= \frac{1}{(2N-1)!!} G_N(0) = F(0) = f(0).$$

Therefore we get

$$(22)\quad (S_R^{(n)}f)(0)=-\frac{\omega_{n-1}}{(\sqrt{2\pi})^n}\sum_{k=1}^{N} f_{k-1}(a)D_R^{(n-2k)}(a)+f(0)+o(1)\quad (R\to\infty).$$

Thus under the condition (5), (4) is valid.

When $n=2N+2$ $(N\geqq 1)$, by (20) we have

$$\frac{d}{dt}f_{N-1}(t)=t\{(2N)!!F(t)+\sum_{1\leqq i,\,j\leqq N} c_{ij}\, t^i\, F^{(j)}(t)\}=tG_N(t)$$

say, and we set $g(x)=G_N(|x|)$. In this case since by (16) we have

$$\frac{\omega_{n-1}}{(\sqrt{2\pi})^n}=\frac{1}{(2N)!!}\,\frac{\omega_1}{(\sqrt{2\pi})^2},$$

so we can rewrite (19) as

$$(23)\quad (S_R^{(n)}f)(0)=-\frac{\omega_{n-1}}{(\sqrt{2\pi})^n}\sum_{k=1}^{N} f_{k-1}(a)D_R^{(n-2k)}(a)$$

$$+\frac{1}{(2N)!!}\,\frac{\omega_1}{(\sqrt{2\pi})^2}\int_0^a D_R^{(2)}(t)tG_N(t)\,dt.$$

The assumption $F\in C^{\ell+2}([0,a])=C^{N+1}([0,a])$ implies $G_N\in C^1([0,a])$. So from (I) with n=2 we have

$$\lim_{R\to\infty}\text{ the second term of (21)}=\frac{1}{(2N)!!}\lim_{R\to\infty}(S_R^{(2)}g)(0)=\frac{1}{(2N)!!}g(0)$$

$$=\frac{1}{(2N)!!}G_N(0)=F(0)=f(0).$$

Therefore the same expression as (22) holds and so (4) is valid under the condition (5). Thus (II) is proved.

We pass to prove (III). We will use (22) which is valid in both of the cases $n=2N+1$ and $n=2N+2$ $(N\geqq 1)$. Suppose that (5) is not satisfied $(\ell=N-1)$ and let

k_0 be in the statement of (Ⅱ). Then by (18) we have $f_{k_0}(\alpha)\neq 0$ and $f_k(\alpha)=0$ for all $k=0,1,\ldots,k_0-1$. So we can write (22) as

$$(24)\quad (S_R{}^{(n)}f)(0)=-\frac{\omega_{n-1}}{(\sqrt{2\pi})^n}\sum_{k=k_0+1}^{N} f_{k-1}(\alpha)D_R{}^{(n-2k)}(\alpha)+f(0)+o(1)\quad (R\to\infty)$$

For each $k=k_0+1, k_0+2,\ldots,N$, we have by (14) and (10),

$$f_{k-1}(\alpha)D_R{}^{(n-2k)}(\alpha)=f_{k-1}(\alpha)\ R^{n-2k}\ V_{(n-2k)/2}(R\alpha)$$

$$=f_{k-1}(\alpha)R^{n-2k}\sqrt{2/\pi}\,(R\alpha)^{-(n-2k)/2-1/2}\{\cos(R\alpha-(n-2k)\pi/4-\pi/4)+O(1/R)\}$$

$$=C(k)f_{k-1}(\alpha)\ R^{(n-1)/2-k}\ \{\sin(R\alpha-(n-2k-1)\pi/4)+O(1/R)\},$$

where $C(k)=\sqrt{2/\pi}\ \alpha^{-(n-2k+1)/2}$ (>0). Especially for $k=k_0+1$ we have

$$(25)\quad f_{k_0}(\alpha)D_R{}^{(n-2(k_0+1))}(\alpha)$$

$$=C(k_0+1)f_{k_0}(\alpha)\ R^{(n-3)/2-k_0}\ \{\sin(R\alpha-(n-2k_0-3)\pi/4)+O(1/R)\}$$

and $C(k_0+1)f_{k_0}(\alpha)\neq 0$. For $k=k_0+2,\ldots,N$, since

$$(n-1)/2-k\leqq(n-1)/2-(k_0+2)=(n-3)/2-k_0-1,$$

so we have

$$(26)\quad f_{k-1}(\alpha)D_R{}^{(n-2k)}(\alpha)=O(R^{(n-1)/2-k})=O(R^{(n-3)/2-k_0-1}).$$

Hence from (24)～(26) we get

$$(S_R{}^{(n)}f)(0)-f(0)$$

$$=C(k_0+1)f_{k_0}(\alpha)\ R^{(n-3)/2-k_0}\ \sin(R\alpha-(n-2k_0-3)\pi/4)+O(R^{(n-3)/2-k_0-1}).$$

Thus (Ⅱ) is valid and Theorem 1 is proved.

§4. The proof of Theorem 2.

First we will prove the theorem in the cases of $n=1$ and 2, and in the case of $n\geqq 3$ we reduce the case to them similarly to Theorem 1. Let $F\in C^2([0,\alpha])$, fix x with $0<|x|\leqq\alpha$ and put $\rho=|x|(>0)$.

When $n=1$, we can write by (11), (12) and (6) as

$$(S_R^{(1)}f)(\rho)=\int_0^R\{\int_0^\alpha F(t)V_{-1/2}(rt)dt\}\,V_{-1/2}(\rho r)dr$$

$$=2/\pi\int_0^R\{\int_0^\alpha F(t)\cos(rt)dt\}\cos(\rho r)dr$$

$$=(\sqrt{2\pi})^{-1}\int_{-R}^R\{(\sqrt{2\pi})^{-1}\int_{-\alpha}^\alpha f(t)\,e^{-iyt}dt\}\,e^{ixy}dy$$

$$=(\sqrt{2\pi})^{-1}\int_{-R}^R\hat{f}(y)\,e^{ixy}dy.$$

Since $f(t)$ $(t\in[-\alpha,\alpha])$ belongs to $L^1(-\alpha,\alpha)$ and is of bounded variation in a neighborhood of the point x, by making use of the Dirichlet-Jordan theorem we have

$$\lim_{R\to\infty}(S_R^{(1)}f)(|x|)=\{f(x+0)+f(x-0)\}/2=\begin{cases}f(x) & (0<|x|<\alpha),\\ f(x)/2 & (|x|=\alpha).\end{cases}$$

So in the case of $n=1$ the theorem is valid.

When $n=2$, we can write by (11) and (12) as

$$(S_R^{(2)}f)(\rho)=\int_0^R\{\int_0^\alpha F(t)tV_0(rt)dt\}\,rV_0(\rho r)dr$$

$$=\int_0^R\{\int_0^\alpha F(t)tJ_0(rt)dt\}\,rJ_0(\rho r)dr$$

$$= \rho^{-1/2} \int_0^R \{ \int_0^a F(t)t^{1/2}(rt)^{1/2}J_0(rt)dt\}(\rho r)^{1/2}J_0(\rho r)dr.$$

We put $F(t)t^{1/2}=G(t)$. Since $G(t)$ belongs to $L^1(0, a)$ and is of bounded variation in a neighborhood of the point $t=\rho\in(0, a]$, by applying Lemma 3 (Hankel inversion formula) with $\mu=0$ we get

$$\lim_{R\to\infty} (S_R^{(2)} f)(\rho)= \rho^{-1/2} \{G(\rho+0)+G(\rho-0)\}/2=\{F(\rho+0)+F(\rho-0)\}/2$$

$$= \begin{cases} F(\rho)=f(x) & (0<|x|<a), \\ F(\rho)/2=f(x)/2 & (|x|=a). \end{cases}$$

Thus in the case of $n=2$ the theorem is proved.

When $n\geqq3$, applying Lemma 2(ii) and integrating by parts in (12) we have

$$(S_R^{(n)} f)(\rho)= \int_0^R \widehat{F}_n(r)\ r^{n-1}\ V_{(n-2)/2}(\rho r)\ dr$$

$$= \int_0^R \{-\frac{1}{r}\frac{d}{dr}\widehat{F}_{n-2}(r)\}\ r^{n-1}\ V_{(n-2)/2}(\rho r)\ dr$$

$$= -\int_0^R \{\frac{d}{dr}\widehat{F}_{n-2}(r)\}\ r^{n-2}\ V_{(n-2)/2}(\rho r)\ dr$$

$$(27) \quad = -\widehat{F}_{n-2}(R)R^{n-2}V_{(n-2)/2}(R\rho)+ \int_0^R \widehat{F}_{n-2}(r)\ \frac{d}{dr}\{r^{n-2}\ V_{(n-2)/2}(\rho r)\}dr.$$

By Lemma 2(i) and (10) we have

$$\text{the first term of } (27)=O(R^{-\{(n-2)+1\}/2}\ R^{n-2}\ R^{-(n-2)/2-1/2})=O(R^{-1})$$

as $R\to\infty$. Since by (8),

$$\frac{d}{dr}\{r^{n-2}\ V_{(n-2)/2}(\rho r)\}= \rho^{-(n-3)}\ \frac{d}{dr}\{(\rho r)^{(n-2)/2}\ J_{(n-2)/2}(\rho r)\}$$

$$= \rho^{-(n-3)}(\rho r)^{(n-2)/2}\ J_{(n-2)/2-1}(\rho r)\rho = r^{n-3}(\rho r)^{-(n-4)/2}\ J_{(n-4)/2}(\rho r)$$

$$=r^{n-3}\, V_{(n-4)/2}(\rho r),$$

so we can write the second term of (27) as

$$\int_0^R \hat{F}_{n-2}(r)\, r^{(n-2)-1}\, V_{(n-2)/2-1}(\rho r)\, dr = (S_R^{(n-2)} f)(\rho).$$

Therefore (27) can be written as

$$(S_R^{(n)} f)(\rho) = (S_R^{(n-2)} f)(\rho) + O(1/R) \quad (R\to\infty).$$

Hence the convergence and the limit of $(S_R^{(n)} f)(\rho)$ are identical with those of $(S_R^{(n-2)} f)(\rho)$. So the cases of odd n and even n are reduced to the cases of $n=1$ and $n=2$ separately. Thus the theorem is proved.

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